

Virasoro vertex operator algebras, the (nonmeromorphic) operator product expansion and the tensor product theory

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Abstract

In [HL1]–[HL5] and [H1], a theory of tensor products of modules for a vertex operator algebra is being developed. To use this theory, one first has to verify that the vertex operator algebra satisfies certain conditions. We show in the present paper that for any vertex operator algebra containing a vertex operator subalgebra isomorphic to a tensor product algebra of minimal Virasoro vertex operator algebras (vertex operator algebras associated to minimal models), the tensor product theory can be applied. In particular, intertwining operators for such a vertex operator algebra satisfy the (nonmeromorphic) commutativity (locality) and the (nonmeromorphic) associativity (operator product expansion). Combined with a result announced in [HL4], the results of the present paper also show that the category of modules for such a vertex operator algebra has a natural structure of a braided tensor category. In particular, for any pair p, q of relatively prime positive integers larger than 1, the category of minimal modules of central charge $1 - 6\frac{(p-q)^2}{pq}$ for the Virasoro algebra has a natural structure of a braided tensor category.

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0 Introduction

Vertex operator algebras were introduced by Borchers [B1] and, in a modified form, by Frenkel, Lepowsky and Meurman [FLM2]. They are essentially the same as chiral algebras in physics (see, for example, [MS]). One crucial property of vertex operator algebras and modules for them is the meromorphicity. It is the meromorphicity that makes it possible to express the axioms of vertex operator algebras using formal series or components of vertex operators. These axioms expressed using formal series or components of vertex operators allow us to construct many examples of vertex operator algebras using algebraic methods. But in conformal field theory and in applications of conformal field theory, the more interesting objects to study are intertwining operators (or chiral vertex operators). In general, intertwining operators do not satisfy the meromorphicity condition since by definition they involve non-integer powers of a variable. The most fundamental assumption on intertwining operators is the so called “operator product expansion” in physics. In the terminology of the theory of vertex operator algebras, the operator product expansion of chiral vertex operators is exactly the associativity of intertwining operators. Many beautiful results in conformal field theory are obtained based on this assumption. One example is the braided tensor category structures constructed from conformal field theories [MS]. In [MS], Moore and Seiberg obtained the braiding and fusing matrices (the commutativity and the associativity isomorphisms) using the assumed (nonmeromorphic) operator product expansion of chiral vertex operators (intertwining operators) or an equivalent geometric axiom of conformal field theories.

Another example is the study of orbifold conformal field theories [DHVW1] [DHVW2] [DFMS] [DVVV]. Orbifold conformal field theories (or simply orbifold theories) are a rich source of conformal field theories. The moonshine module constructed by Frenkel, Lepowsky and Meurman [FLM1] [FLM2] is historically the first example of orbifold theories. Dolan, Goddard and Montague gave another proof that the moonshine module is a “meromorphic conformal field theory” (vertex operator algebra) using techniques developed in string theory [DGM1]. Their proof works for a class of \mathbb{Z}_2 -orbifold theories and thus allows them to obtain a better understanding about Frenkel-Lepowsky-Meurman’s triality [DGM2] [DGM3]. Another orbifold construction—a \mathbb{Z}_3 -orbifold construction—of the moonshine module was given by Dong and Mason [DM1]. In [T1] and [T2], Tuite showed that the

Monstrous moonshine conjectured by Conway and Norton [CN] and proved by Borchers [B2] for Frenkel-Lepowsky-Meurman's moonshine module can be understood using the uniqueness, conjectured by Frenkel, Lepowsky and Meurman [FLM2], of the moonshine module and some conjectures on the orbifold theories constructed from the moonshine module. Orbifold theories were also used to construct the mirror of a Calabi-Yau manifold [GP]. In all of the studies of orbifold theories mentioned above, the results are either only about meromorphic parts or based on the basic assumption that the nonmeromorphic operator product expansion holds.

To establish conformal field theory as a solid mathematical theory and to solve concrete mathematical problems using conformal field theories, we need to prove, not assume, the associativity (operator product expansion) of intertwining operators (chiral vertex operators). The main difficulty—but also the source of the richness of conformal field theories—is that products and iterates of intertwining operators are in general not meromorphic. One way to do this is to construct the intertwining operators directly from relatively elementary mathematical data and to then show that they satisfy associativity. This method seems to be very difficult and is also not practical since one has to construct intertwining operators and to prove the associativity case by case. Another method, the method that we use in the present and related papers, is to study intertwining operators as objects in the representation theory of vertex operator algebras.

In the representation theory of Lie algebras, one of the most important operations is the tensor product operation for modules for Lie algebras. In the representation theory of vertex operator algebras, a tensor product theory for modules for a vertex operator algebra is being developed in [HL1]–[HL5] and [H1]. But to use this theory, one first has to verify that the vertex operator algebra satisfies certain conditions. The purpose of the present paper is to show that for any vertex operator algebra containing a vertex operator subalgebra isomorphic to a tensor product algebra of minimal Virasoro vertex operator algebras (vertex operator algebras associated minimal models), the tensor product theory can be applied. In particular, intertwining operators for such a vertex operator algebra satisfy the (nonmeromorphic) commutativity (locality) and the (nonmeromorphic) associativity (operator product expansion). Recall that a vertex operator subalgebra of a vertex operator algebra is a subspace which contains the vacuum vector and the Virasoro element and is closed under the vertex operator map [FHL]. The condition that

a vertex operator algebra contains a vertex operator subalgebra isomorphic to a tensor product algebra of minimal Virasoro vertex operator algebras gives enough information we need about the vertex operator algebra. For example, if the vertex operator algebra contains a trivial one-dimensional vertex operator subalgebra isomorphic to the null tensor product of minimal Virasoro vertex operator algebras, then the vertex operator algebra is a finite-dimensional commutative associative algebra.

Combined with a result announced in [HL4], the results of the present paper also show that the category of modules for such a vertex operator algebra has a natural structure of a braided tensor category. In particular, for any pair p, q of relatively prime positive integers larger than 1, the category of minimal modules of central charge $1 - 6\frac{(p-q)^2}{pq}$ for the Virasoro algebra has a natural structure of a braided tensor category. Since in general the vertex operator algebras studied in this paper are not rational (see Example 3.4), we also relax the conditions to use the tensor product theory given in [HL2] [HL4] [HL5] and [H1], especially the rationality of the vertex operator algebra, in this paper. For the precise statements of the results in the present paper, see Section 3.

The results of the present paper have many applications. One immediate consequence is that for many vertex operator algebras associated to \mathcal{W} -algebras, the tensor product theory can be applied. The results of the present paper have been used in the proof by the author that there is a natural structure of an abelian intertwining algebra (in the sense of Dong and Lepowsky [DL]) on the direct sum of the untwisted vertex operator algebra constructed from the Leech lattice and its (unique) irreducible twisted module (see [H2]). A special case of this abelian intertwining algebra structure gives a new and conceptual proof that the moonshine module is a vertex operator algebra. This abelian intertwining algebra also contains as substructures several other interesting structures, including a twisted module for the moonshine module, the superconformal structure of Dixon, Ginsparg and Harvey [DGH], a vertex operator superalgebra and a twisted module for it. We believe that the results of the present paper can also be used to give a new and conceptual proof of the result of Dolan, Goddard and Montague [DGM1]. The results of the present paper can also be reformulated in terms of (partial) operads, and genus-zero conformal field theories in the sense of Segal [S] can be constructed using this reformulation. For vertex operator

algebras containing as vertex operator subalgebras tensor product algebras of vertex operator algebras associated to affine Lie algebras, we have similar results. All of these will be discussed in separate papers.

This paper is organized as follows: In Section 1, we review briefly the tensor product theory for modules for a vertex operator algebra developed in [HL1]–[HL5] and [H1]. In Section 2, we review the results on the minimal Virasoro vertex operator algebras and their modules obtained in [FZ], [DMZ] and [W]. We state the results of the present paper, including Theorem 3.1, Theorem 3.2, Theorem 3.5, Proposition 3.7, Theorem 3.8, Corollary 3.9 and Theorem 3.10, in Section 3. In Section 3, we also give an example showing that for any rational vertex operator algebra, there exists an irrational vertex operator algebra containing this rational vertex operator algebra as a vertex operator subalgebra. Theorem 3.1, Theorem 3.2, Theorem 3.5 and Proposition 3.7 are proved in Section 4, Section 5, Section 6 and Section 7, respectively.

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1 The tensor product theory

We review briefly in this section the tensor product theory for modules for a vertex operator algebra being developed in [HL1]–[HL5] and [H1].

Fix $z \in \mathbb{C}^\times$ and let (W_1, Y_1) , (W_2, Y_2) and (W_3, Y_3) be V -modules. A $P(z)$ -intertwining map of type $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$ is a linear map $F : W_1 \otimes W_2 \rightarrow \overline{W}_3$ satisfying the condition

$$\begin{aligned} x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) Y_3(v, x_1) F(w_{(1)} \otimes w_{(2)}) &= \\ &= z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) F(Y_1(v, x_0) w_{(1)} \otimes w_{(2)}) \\ &\quad + x_0^{-1} \delta \left(\frac{z - x_1}{-x_0} \right) F(w_{(1)} \otimes Y_2(v, x_1) w_{(2)}) \end{aligned}$$

for $v \in V$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$. The vector space of $P(z)$ -intertwining maps of type $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$ is denoted by $\mathcal{M}[P(z)]_{W_1 W_2}^{W_3}$. A $P(z)$ -product of W_1 and W_2 is a V -module (W_3, Y_3) equipped with a $P(z)$ -intertwining map F of type $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$ and is denoted by $(W_3, Y_3; F)$ (or simply by (W_3, F)). Let $(W_4, Y_4; G)$

be another $P(z)$ -product of W_1 and W_2 . A *morphism* from $(W_3, Y_3; F)$ to $(W_4, Y_4; G)$ is a module map η from W_3 to W_4 such that $G = \bar{\eta} \circ F$, where $\bar{\eta}$ is the map from \overline{W}_3 to \overline{W}_4 uniquely extending η .

A $P(z)$ -*tensor product* of W_1 and W_2 is a $P(z)$ -product

$$(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$$

such that for any $P(z)$ -product $(W_3, Y_3; F)$, there is a unique morphism from

$$(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$$

to $(W_3, Y_3; F)$. The V -module $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)})$ is called a $P(z)$ -*tensor product module* of W_1 and W_2 . A $P(z)$ -tensor product is unique up to isomorphism.

To construct a $P(z)$ -tensor product of W_1 and W_2 , we define an action of

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$$

on $(W_1 \otimes W_2)^*$ (where ι_+ denotes the operation of expansion of a rational function of t in the direction of positive powers of t), that is, a linear map

$$\tau_{P(z)} : V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}] \rightarrow \text{End } (W_1 \otimes W_2)^*,$$

by

$$\begin{aligned} & \left(\tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) \lambda \right) (w_{(1)} \otimes w_{(2)}) \\ &= z^{-1} \delta \left(\frac{x_1^{-1} - x_0}{z} \right) \lambda(Y_1(e^{x_1 L(1)} (-x_1^{-2})^{L(0)} v, x_0) w_{(1)} \otimes w_{(2)}) \\ & \quad + x_0^{-1} \delta \left(\frac{z - x_1^{-1}}{-x_0} \right) \lambda(w_{(1)} \otimes Y_2^*(v, x_1) w_{(2)}) \end{aligned}$$

for $v \in V$, $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, where

$$Y_t(v, x) = v \otimes x^{-1} \delta \left(\frac{t}{x} \right).$$

There is an obvious action of

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$$

on any V -module. We have:

Proposition 1.1 *Under the natural isomorphism*

$$\mathrm{Hom}(W'_3, (W_1 \otimes W_2)^*) \xrightarrow{\sim} \mathrm{Hom}(W_1 \otimes W_2, \overline{W}_3),$$

the maps in $\mathrm{Hom}(W'_3, (W_1 \otimes W_2)^)$ intertwining the two actions of*

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$$

on W'_3 and $(W_1 \otimes W_2)^$ correspond exactly to the $P(z)$ -intertwining maps of type $\binom{W_3}{W_1 W_2}$.*

Write

$$Y'_{P(z)}(v, x) = \tau_{P(z)}(Y_t(v, x))$$

and

$$Y'_{P(z)}(\omega, x) = \sum_{n \in \mathbb{Z}} L'_{P(z)}(n) x^{-n-2}.$$

We call the eigenspaces of the operator $L'_{P(z)}(0)$ the *weight subspaces* or *homogeneous subspaces* of $(W_1 \otimes W_2)^*$, and we have the corresponding notions of *weight vector* (or *homogeneous vector*) and *weight*.

Let W be a subspace of $(W_1 \otimes W_2)^*$. We say that W is *compatible* for $\tau_{P(z)}$ if every element of W satisfies the following nontrivial and subtle condition (called the *compatibility condition*) on $\lambda \in (W_1 \otimes W_2)^*$: The formal Laurent series $Y'_{P(z)}(v, x_0)\lambda$ involves only finitely many negative powers of x_0 and

$$\begin{aligned} \tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) \lambda = \\ = x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y'_{P(z)}(v, x_1) \lambda \quad \text{for all } v \in V. \end{aligned}$$

Also, we say that W is (\mathbb{C}) -*graded* if it is \mathbb{C} -graded by its weight subspaces, and that W is a V -*module* (respectively, *generalized module*) if W is graded and is a module (respectively, generalized module, see [HL1] and [HL2]) when equipped with this grading and with the action of $Y'_{P(z)}(\cdot, x)$. The weight subspace of a subspace W with weight $n \in \mathbb{C}$ will be denoted $W_{(n)}$.

Define

$$W_1 \boxtimes_{P(z)} W_2 = \sum_{W \in \mathcal{W}_{P(z)}} W = \bigcup_{W \in \mathcal{W}_{P(z)}} W \subset (W_1 \otimes W_2)^*,$$

where $\mathcal{W}_{P(z)}$ is the set of all compatible modules for $\tau_{P(z)}$ in $(W_1 \otimes W_2)^*$.

We need:

Definition 1.2 A vertex operator algebra V is *rational* if it satisfies the following conditions:

1. There are only finitely many irreducible V -modules (up to equivalence).
2. Every V -module is completely reducible (and is in particular a *finite* direct sum of irreducible modules).
3. All the fusion rules (the dimensions of spaces of intertwining operators) for V are finite (for triples of irreducible modules and hence arbitrary modules).

We have:

Proposition 1.3 *Let V be a rational vertex operator algebra and W_1, W_2 V -modules. Then $(W_1 \boxtimes_{P(z)} W_2, Y'_{P(z)}|_{V \otimes W_1 \boxtimes_{P(z)} W_2})$ is a module.*

If $W_1 \boxtimes_{P(z)} W_2$ is a module, we define a V -module $W_1 \boxtimes_{P(z)} W_2$ by

$$W_1 \boxtimes_{P(z)} W_2 = (W_1 \boxtimes_{P(z)} W_2)'$$

and we write the corresponding action as $Y_{P(z)}$. Applying Proposition 1.1 to the special module $W_3 = W_1 \boxtimes_{P(z)} W_2$ and the identity map $W'_3 \rightarrow W_1 \boxtimes_{P(z)} W_2$, we obtain a canonical $P(z)$ -intertwining map of type $\begin{pmatrix} W_1 \boxtimes_{P(z)} W_2 \\ W_1 \quad W_2 \end{pmatrix}$, which we denote

$$\begin{aligned} \boxtimes_{P(z)} : W_1 \otimes W_2 &\rightarrow \overline{W_1 \boxtimes_{P(z)} W_2} \\ w_{(1)} \otimes w_{(2)} &\mapsto w_{(1)} \boxtimes_{P(z)} w_{(2)}. \end{aligned}$$

We have:

Proposition 1.4 *Assume that $W_1 \boxtimes_{P(z)} W_2$ is a module. Then the $P(z)$ -product $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$ is a $P(z)$ -tensor product of W_1 and W_2 .*

Combining this proposition with Proposition 1.3, we obtain:

Proposition 1.5 *Assume that V is rational. Then $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$ is a $P(z)$ -tensor product of W_1 and W_2 .*

Observe that any element of $W_1 \boxtimes_{P(z)} W_2$ is an element λ of $(W_1 \otimes W_2)^*$ satisfying:

The compatibility condition

- (a) The *lower truncation condition*: For all $v \in V$, the formal Laurent series $Y'_{P(z)}(v, x)\lambda$ involves only finitely many negative powers of x .
- (b) The formula (1.1) holds.

The local grading-restriction condition

- (a) The *grading condition*: λ is a (finite) sum of weight vectors of $(W_1 \otimes W_2)^*$.
- (b) Let W_λ be the smallest subspace of $(W_1 \otimes W_2)^*$ containing λ and stable under the component operators $\tau_{P(z)}(v \otimes t^n)$ of the operators $Y'_{P(z)}(v, x)$ for $v \in V$, $n \in \mathbb{Z}$. Then the weight spaces $(W_\lambda)_{(n)}$, $n \in \mathbb{C}$, of the (graded) space W_λ have the properties

$$\begin{aligned} \dim (W_\lambda)_{(n)} &< \infty \quad \text{for } n \in \mathbb{C}, \\ (W_\lambda)_{(n)} &= 0 \quad \text{for } n \text{ whose real part is sufficiently small.} \end{aligned}$$

We have another construction of $W_1 \boxtimes_{P(z)} W_2$ using these conditions:

Theorem 1.6 *The subspace of $(W_1 \otimes W_2)^*$ consisting of the elements satisfying the compatibility condition and the local grading-restriction condition, equipped with $Y'_{P(z)}$, is a generalized module and is equal to $W_1 \boxtimes_{P(z)} W_2$.*

The following results follows immediately from the theorem above, the definition of $W_1 \boxtimes_{P(z)} W_2$ and Proposition 1.3:

Corollary 1.7 *Assume that $W_1 \boxtimes_{P(z)} W_2$ is a module. Then the contragredient module of the module $W_1 \boxtimes_{P(z)} W_2$, equipped with the $P(z)$ -intertwining map $\boxtimes_{P(z)}$, is a $P(z)$ -tensor product of W_1 and W_2 equal to the structure $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$ constructed above.*

Corollary 1.8 *Assume that V is a rational vertex operator algebra. Then the contragredient module of the module $W_1 \boxtimes_{P(z)} W_2$, equipped with the $P(z)$ -intertwining map $\boxtimes_{P(z)}$, is a $P(z)$ -tensor product of W_1 and W_2 equal to the structure $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$ constructed above.*

The rationality of a vertex operator algebra is sufficient for the construction of the tensor product of two modules for this vertex operator algebra, but is still not enough to guarantee that the tensor products satisfy a certain associativity property and that the category of modules is a vertex tensor category defined in [HL4]. So we need more conditions.

We recall these other conditions. Assume that V is rational and all irreducible V -modules are \mathbb{R} -graded. Note that in this case all V -modules are \mathbb{R} -graded, that is, weights of elements of V -modules are always real numbers and that all intertwining operators are formal series of real powers. Given any V -modules W_1, W_2, W_3, W_4 and W_5 , let $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3$ and \mathcal{Y}_4 be intertwining operators of type $\begin{pmatrix} W_4 \\ W_1 W_5 \end{pmatrix}, \begin{pmatrix} W_5 \\ W_2 W_3 \end{pmatrix}, \begin{pmatrix} W_5 \\ W_1 W_2 \end{pmatrix}$ and $\begin{pmatrix} W_4 \\ W_5 W_3 \end{pmatrix}$, respectively. For any V -module W , let $P_n, n \in \mathbb{R}$, be the projection from W to its homogeneous $W_{(n)}$ subspace of weight n . For any $w_{(1)} \in W_1, w_{(2)} \in W_2, w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$, we say that

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_2) \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)} \rangle_{W_4} \Big|_{x_1=z_1, x_2=z_2} \quad (1.1)$$

is *convergent* if the series

$$\sum_{n \in \mathbb{R}} \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_2) P_n(\mathcal{Y}_2(w_{(2)}, x_2) w_{(3)}) \rangle_{W_4} \Big|_{x_1=z_1, x_2=z_2}$$

is absolutely convergent for any choices of $\log z_1$ and $\log z_2$ in the definitions of $z_1^n = e^{n \log z_1}$ and $z_2^n = e^{n \log z_2}$, $n \in \mathbb{R}$. Similarly, we say that

$$\langle w'_{(4)}, \mathcal{Y}_4(\mathcal{Y}_3(w_{(1)}, x_0) w_{(2)}, x_2) w_{(3)} \rangle_{W_4} \Big|_{x_0=z_1-z_2, x_2=z_2} \quad (1.2)$$

is *convergent* if the series

$$\sum_{n \in \mathbb{R}} \langle w'_{(4)}, \mathcal{Y}_4(P_n(\mathcal{Y}_3(w_{(1)}, x_0) w_{(2)}), x_2) w_{(3)} \rangle_{W_4} \Big|_{x_0=z_1-z_2, x_2=z_2}$$

is absolutely convergent for any choices of $\log(z_1 - z_2)$ and $\log z_2$ in the definitions of $(z_1 - z_2)^n = e^{n \log(z_1 - z_2)}$ and $z_2^n = e^{n \log z_2}$, $n \in \mathbb{R}$. Consider the following conditions for the product of \mathcal{Y}_1 and \mathcal{Y}_2 and for the iterate of \mathcal{Y}_3 and \mathcal{Y}_4 , respectively:

Convergence and extension property for products There exists an integer N (depending only on \mathcal{Y}_1 and \mathcal{Y}_2), and for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$, $w'_{(4)} \in W'_4$, there exist $j \in \mathbb{N}$, $r_i, s_i \in \mathbb{R}$, $i = 1, \dots, j$, and analytic functions $f_i(z)$ on $|z| < 1$, $i = 1, \dots, j$, satisfying

$$\text{wt } w_{(1)} + \text{wt } w_{(2)} + s_i > N, \quad i = 1, \dots, j, \quad (1.3)$$

such that

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_2) \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)} \rangle_{W_4} \Big|_{x_1=z_1, x_2=z_2}$$

is convergent when $|z_1| > |z_2| > 0$ and can be analytically extended to the multi-valued analytic function

$$\sum_{i=1}^j z_2^{r_i} (z_1 - z_2)^{s_i} f_i \left(\frac{z_1 - z_2}{z_2} \right) \quad (1.4)$$

when $|z_2| > |z_1 - z_2| > 0$.

Convergence and extension property for iterates There exists an integer \tilde{N} (depending only on \mathcal{Y}_3 and \mathcal{Y}_4), and for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$, $w'_{(4)} \in W'_4$, there exist $k \in \mathbb{N}$, $\tilde{r}_i, \tilde{s}_i \in \mathbb{R}$, $i = 1, \dots, k$, and analytic functions $\tilde{f}_i(z)$ on $|z| < 1$, $i = 1, \dots, k$, satisfying

$$\text{wt } w_{(2)} + \text{wt } w_{(3)} + \tilde{s}_i > \tilde{N}, \quad i = 1, \dots, k,$$

such that

$$\langle w'_{(4)}, \mathcal{Y}_4(\mathcal{Y}_3(w_{(1)}, x_0) w_{(2)}, x_2) w_{(3)} \rangle_{W_4} \Big|_{x_0=z_1-z_2, x_2=z_2}$$

is convergent when $|z_2| > |z_1 - z_2| > 0$ and can be analytically extended to the multi-valued analytic function

$$\sum_{i=1}^k z_1^{\tilde{r}_i} z_2^{\tilde{s}_i} \tilde{f}_i \left(\frac{z_2}{z_1} \right)$$

when $|z_1| > |z_2| > 0$.

If for any V -modules W_1, W_2, W_3, W_4 and W_5 and any intertwining operators \mathcal{Y}_1 and \mathcal{Y}_2 of the types as above, the convergence and extension property for products holds, we say that *the products of the intertwining operators for V have the convergence and extension property*. Similarly we can define the meaning of the phrase *the iterates of the intertwining operators for V have the convergence and extension property*.

We also need a condition on certain generalized modules. If a generalized V -module $W = \coprod_{n \in \mathbb{C}} W_{(n)}$ satisfying the condition that $W_{(n)} = 0$ for n whose real part is sufficiently small, we say that W is *lower-truncated*. Another condition that we need to establish the associativity is that every finitely-generated lower-truncated generalized V -module is a V -module.

Assume that for any V -modules $W_i, i = 1, \dots, 5$, any intertwining operators $\mathcal{Y}_1, \mathcal{Y}_2$ of the type above, (1.1) is convergent for all $w_{(1)} \in W_1, w_{(2)} \in W_2, w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$, and for any V -modules $W_i, i = 1, \dots, 5$, any intertwining operators $\mathcal{Y}_3, \mathcal{Y}_4$ of the type above, (1.2) is convergent for all $w_{(1)} \in W_1, w_{(2)} \in W_2, w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$. Let W_1, W_2 and W_3 be three V -modules, $w_{(1)} \in W_1, w_{(2)} \in W_2$ and $w_{(3)} \in W_3$ and $z_1, z_2 \in \mathbb{C}$ satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$. From [HL2], we know that any $P(z)$ -intertwining maps (for $z = z_1, z_2, z_1 - z_2$) can be obtained from certain intertwining operators by substituting complex powers of $e^{\log z}$ for the complex powers of the formal variable x . Thus $w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})$ (or $(w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)}$) is a product (or an iterate) of two intertwining operators evaluated at $w_{(1)} \otimes w_{(2)} \otimes w_{(3)}$ and with the complex powers of the formal variables replaced by the complex powers of $e^{\log z_1}$ and of $e^{\log z_2}$ (or by the complex powers of $e^{\log(z_1-z_2)}$ and of $e^{\log z_2}$). By assumption, $w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})$ (or $(w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)}$) is a well-defined element of $\overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)}$ (or of $\overline{(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3}$). The following associativity of the $P(\cdot)$ -tensor products is proved in [H1]:

Theorem 1.9 *Assume that V is a rational vertex operator algebra and all irreducible V -modules are \mathbb{R} -graded. Also assume that V satisfies the following conditions:*

1. *Every finitely-generated lower-truncated generalized V -module is a V -module.*
2. *The products or the iterates of the intertwining operators for V have the convergence and extension property.*

Then for any V -module W_1, W_2 and W_3 and any complex numbers z_1 and z_2 satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$, there exists a unique isomorphism $\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)}$ from $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$ to $(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$ such that for any $w_{(1)} \in W_1, w_{(2)} \in W_2$ and $w_{(3)} \in W_3$,

$$\begin{aligned} \mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)}(w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})) \\ = (w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)}, \end{aligned}$$

where

$$\overline{\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)}} : \overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)} \rightarrow \overline{(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3}$$

is the unique extension of $\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)}$.

In [H1], it is also proved that the above associativity of the $P(\cdot)$ -tensor product is equivalent to the associativity of the intertwining operators. So we also have the following:

Theorem 1.10 *Let V be a vertex operator algebra satisfying the conditions in Theorem 1.9. Then the intertwining operators for V have the following two associativity properties:*

1. *For any modules W_1, W_2, W_3, W_4 and W_5 and any intertwining operators \mathcal{Y}_1 and \mathcal{Y}_2 of type $\begin{pmatrix} W_4 \\ W_1 W_5 \end{pmatrix}$ and $\begin{pmatrix} W_5 \\ W_2 W_3 \end{pmatrix}$, respectively, there exist a module W_6 and intertwining operators \mathcal{Y}_3 and \mathcal{Y}_4 of type $\begin{pmatrix} W_6 \\ W_1 W_2 \end{pmatrix}$ and $\begin{pmatrix} W_4 \\ W_6 W_3 \end{pmatrix}$, respectively, such that for any $z_1, z_2 \in \mathbb{C}$ satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$,*

$$\begin{aligned} \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)} \rangle_{W_4} \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}} \\ = \langle w'_{(4)}, \mathcal{Y}_4(\mathcal{Y}_3(w_{(1)}, x_0) w_{(2)}, x_2) w_{(3)} \rangle_{W_4} \Big|_{x_0^n = e^{n \log(z_1 - z_2)}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}}. \end{aligned} \tag{1.5}$$

holds for any $w'_{(4)} \in W'_4, w_{(1)} \in W_1, w_{(2)} \in W_2$ and $w_{(3)} \in W_3$.

2. For any modules W_1, W_2, W_3, W_4 and W_6 and any intertwining operators \mathcal{Y}_3 and \mathcal{Y}_4 of type $\begin{pmatrix} W_6 \\ W_1 W_2 \end{pmatrix}$ and $\begin{pmatrix} W_4 \\ W_6 W_3 \end{pmatrix}$, respectively, there exist a module W_5 and intertwining operators \mathcal{Y}_1 and \mathcal{Y}_2 of type $\begin{pmatrix} W_4 \\ W_1 W_5 \end{pmatrix}$ and $\begin{pmatrix} W_5 \\ W_2 W_3 \end{pmatrix}$, respectively, such that for any $z_1, z_2 \in \mathbb{C}$ satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$, (1.5) holds for any $w'_{(4)} \in W'_4$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$.

The result below is a consequence of the main result announced in [HL4] and Theorem 1.10. It gives the structure of a vertex tensor category to the category of modules for a vertex operator algebra satisfying the conditions discussed above and an additional convergence condition. See [HL4] for the definitions of \tilde{K}^c , ψ and vertex tensor category.

Theorem 1.11 *Let V be a vertex operator algebra of central charge c satisfying the conditions in Theorem 1.9. We assume in addition that V satisfies the following condition:*

3. For any V -modules W_j , $j = 1, \dots, 2m+1$, any intertwining operators \mathcal{Y}_i , $i = 1, \dots, m$, of types $\begin{pmatrix} W_{2i-1} \\ W_{2i} W_{2i+1} \end{pmatrix}$, respectively, and any $w'_{(1)} \in W'_1$, $w_{(2i)} \in W_{2i}$, $i = 1, \dots, m$, and $w_{(2m+1)} \in W_{2m+1}$,

$$\left\langle w'_1, \mathcal{Y}_1(w_{(2)}, x_1) \cdots \mathcal{Y}_m(w_{(2m)}, x_m) w_{(2m+1)} \right\rangle \Big|_{x_i^n = e^{n \log z_i}, 1 \leq i \leq m, n \in \mathbb{C}}$$

is absolutely convergent for any $z_1, \dots, z_n \in \mathbb{C}$ satisfying $|z_1| > \cdots > |z_n| > 0$.

Then the category of V -modules has a natural structure of a vertex tensor category of central charge c such that for each $z \in \mathbb{C}^\times$, the tensor product bifunctor $\boxtimes_{\psi(P(z))}$ associated to $\psi(P(z)) \in \tilde{K}^c(2)$ is equal to $\boxtimes_{P(z)}$.

The following result is also announced in [HL4] (see [JS] for the definition of braided tensor (monoidal) category):

Theorem 1.12 *The underlying category of a vertex tensor category has a natural structure of a braided tensor category.*

2 Minimal Virasoro vertex operator algebras and modules

In this section, we summarize the results on minimal Virasoro vertex operator algebras obtained in [FZ], [DMZ] and [W].

Let

$$\mathfrak{L} = \oplus_{n \in \mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}d$$

with the commutation relations

$$\begin{aligned} [L_m, L_n] &= (m - n)L_{m+n} + \frac{m^3 - m}{12} \delta_{m+n,0} d, \quad m, n \in \mathbb{Z} \\ [L_m, d] &= 0, \quad m \in \mathbb{Z}, \end{aligned}$$

be the Virasoro algebra. Consider the two subalgebras

$$\begin{aligned} \mathfrak{L}_+ &= \oplus_{n \in \mathbb{Z}_+} \mathbb{C}L_n, \\ \mathfrak{L}_- &= \oplus_{n \in -\mathbb{Z}_+} \mathbb{C}L_n. \end{aligned}$$

Let $U(\cdot)$ be the functor from the category of Lie algebras to the category of associative algebras obtained by taking the universal enveloping algebras of Lie algebras. For any representation of \mathfrak{L} and any $m \in \mathbb{Z}$, we shall use $L(m)$ to denote the representation image of L_m . For any $h, c \in \mathbb{C}$, the Verma module $M(c, h)$ for \mathfrak{L} is a free $U(\mathfrak{L}_-)$ -module generated by $\mathbf{1}_{c,h}$ such that

$$\begin{aligned} \mathfrak{L}_+ \mathbf{1}_{c,h} &= 0, \\ L(0) \mathbf{1}_{c,h} &= h \mathbf{1}_{c,h}, \\ d \mathbf{1}_{c,h} &= c \mathbf{1}_{c,h}. \end{aligned}$$

There exists a unique maximal proper submodule $J(c, h)$ of $M(c, h)$. It is easy to see that if $c \neq 0$, then both $\mathbf{1}_{c,0}$ and $L(-2)\mathbf{1}_{c,0}$ are not in $J(c, 0)$. Let

$$L(c, h) = M(c, h)/J(c, h).$$

Then $L(c, 0)$ has the structure of a vertex operator algebra with vacuum $\mathbf{1}_{c,0}$ and the Virasoro element $L(-2)\mathbf{1}_{c,0}$ [FZ]. The following result is proved in [W] using the results of Feigin and Fuchs on representations of the Virasoro algebra [FF1] [FF2]:

Theorem 2.1 *The vertex operator algebra $L(c, 0)$ is rational if and only if either $c = 0$ or there is a pair p, q of relatively prime positive integers larger than 1 such that*

$$c = c_{p,q} = 1 - 6 \frac{(p-q)^2}{pq}.$$

A set of representatives of the equivalence classes of irreducible modules for $L(c_{p,q}, 0)$ is

$$\{L(c_{p,q}, h_{p,q}^{m,n})\}_{0 < m < p, 0 < n < q, m, n \in \mathbb{Z}}$$

where for any $m, n \in \mathbb{Z}$ satisfying $0 < m < p, 0 < n < q$,

$$h_{p,q}^{m,n} = \frac{(np - mq)^2 - (p - q)^2}{4pq}.$$

For any $m, m', m'', n, n', n'' \in \mathbb{Z}$ satisfying $0 < m, m', m'' < p, 0 < n, n', n'' < q$, the fusion rule $\mathcal{N}_{L(c_{p,q}, h_{p,q}^{m',n'})}^{L(c_{p,q}, h_{p,q}^{m,n})} L(c_{p,q}, h_{p,q}^{m'',n''})$ is 1 if $m + m' + m'' < 2p$, $n + n' + n'' < 2q$, $m < m' + m''$, $m' < m'' + m$, $m'' < m + m'$, $n < n' + n''$, $n' < n'' + n$, $n'' < n + n'$ and the sums $m + m' + m''$, $n + n' + n''$ are odd, and is 0 otherwise.

The rationality of $L(c_{p,p+1}, 0)$ for an integer $p > 1$ was proved first in [DMZ] and the fusion rules in the case $p = 3, q = 4$ were also calculated there.

For any pair p, q of relatively prime positive integers larger than 1, we call the vertex operator algebra $L(c_{p,q}, 0)$ a *minimal Virasoro vertex operator algebra*.

Let n be a positive integer, $(p_i, q_i), i = 1, \dots, n$, n pairs of relatively prime positive integers larger than 1, $V = L(c_{p_1, q_1}, 0) \otimes \dots \otimes L(c_{p_n, q_n}, 0)$. From the results proved in [FHL] and [DMZ], V is a rational vertex operator algebra, a set of representatives of equivalence classes of irreducible modules for V can be given explicitly and the fusion rules for V can be calculated easily.

3 The results of the present paper

First we have the following commutativity for intertwining operators:

Theorem 3.1 *Let V be a vertex operator algebra satisfying the conditions in Theorem 1.9. Then for any modules W_1, W_2, W_3, W_4 and W_5 and any intertwining operators \mathcal{Y}_1 and \mathcal{Y}_2 of type $\left(\begin{smallmatrix} W_4 \\ W_1 W_5 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} W_5 \\ W_2 W_3 \end{smallmatrix}\right)$, respectively, there exist a module W_6 and intertwining operators \mathcal{Y}_3 and \mathcal{Y}_4 of type $\left(\begin{smallmatrix} W_4 \\ W_2 W_6 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} W_6 \\ W_1 W_3 \end{smallmatrix}\right)$, respectively, such that for any $w'_{(4)} \in W'_4$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$, the multi-valued analytic function*

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)} \rangle_{W_4} \Big|_{x_1=z_1, x_2=z_2}$$

of z_1 and z_2 in the region $|z_1| > |z_2| > 0$ and the multi-valued analytic function

$$\langle w'_{(4)}, \mathcal{Y}_3(w_{(2)}, x_2) \mathcal{Y}_4(w_{(1)}, x_1) w_{(3)} \rangle_{W_4} \Big|_{x_1=z_1, x_2=z_2}$$

of z_1 and z_2 in the region $|z_2| > |z_1| > 0$, are analytic extensions of each other.

The next result relaxes the conditions to use the tensor product theory:

Theorem 3.2 *Let V be a vertex operator algebra containing a rational vertex operator subalgebra V_0 . Then for any V -modules W_1 and W_2 , $W_1 \boxtimes_{P(z)} W_2$ is a V -module and the conclusions of Proposition 1.4 and Corollary 1.7 are true. Suppose in addition that we assume that all irreducible V_0 -modules are \mathbb{R} -graded and V and V_0 satisfies the following conditions:*

1. *Every finitely-generated lower-truncated generalized V -module is a V -module.*
2. *The products or the iterates of the intertwining operators for V_0 have the convergence and extension property.*

Then the conclusions of Theorem 1.9, Theorem 1.10 and Theorem 3.1 are true. Suppose that we assume further that V_0 satisfies the following condition:

3. *For any V_0 -modules W_j , $j = 1, \dots, 2m+1$, any intertwining operators \mathcal{Y}_i , $i = 1, \dots, m$, of types $\left(\begin{smallmatrix} W_{2i-1} \\ W_{2i} W_{2i+1} \end{smallmatrix}\right)$, respectively, and any $w'_{(1)} \in W'_1$, $w_{(2i)} \in W_{2i}$, $i = 1, \dots, m$, and $w_{(2m+1)} \in W_{2m+1}$,*

$$\langle w'_{(1)}, \mathcal{Y}_1(w_{(2)}, x_1) \cdots \mathcal{Y}_m(w_{(2m)}, x_m) w_{(2m+1)} \rangle \Big|_{x_i^n = e^{n \log z_i}, 1 \leq i \leq m, n \in \mathbb{C}}$$

is absolutely convergent for any $z_1, \dots, z_n \in \mathbb{C}$ satisfying $|z_1| > \dots > |z_n| > 0$.

Then the conclusion of Theorem 1.11 is true.

By Theorem 3.2 and Theorem 1.12, we obtain:

Corollary 3.3 *Let V be a vertex operator algebra satisfying all conditions in Theorem 3.2. Then the category of V -modules has a natural structure of a braided tensor category.*

Because of the above theorem, It is natural to ask whether a vertex operator algebra containing a rational vertex operator subalgebra is also rational. The answer is no. Here is a simple example:

Example 3.4 Let $(V_0, Y_{V_0}, \mathbf{1}, \omega)$ be a rational vertex operator algebra. Let $V = V_0 \oplus V_0$. Then V is \mathbb{Z} -graded and satisfies the two grading axioms for vertex operator algebras. We denote an element of V by (u, v) where $u, v \in V_0$. We define a vertex operator map $Y_V : V \otimes V \rightarrow V[[x, x^{-1}]]$ by

$$Y_V((u_1, v_1), x)(u_2, v_2) = (Y_{V_0}(u_1, x)u_2, Y_{V_0}(u_1, x)v_2 + Y_{V_0}(v_1, x)u_2)$$

for any $(u_1, v_1), (u_2, v_2) \in V$. Then using the vertex operator algebra structure on V_0 , we see that $(V, Y_V, (\mathbf{1}, 0), (\omega, 0))$ is a vertex operator algebra containing a vertex operator subalgebra isomorphic to V_0 . Let W_0 be the subspace of V consists of all elements of the form $(0, v)$, $v \in V_0$. We define a vertex operator map $Y_{W_0} : V \otimes W_0 \rightarrow W_0[[x, x^{-1}]]$ to be the restriction of Y_V to $V \otimes W_0$. Then it is clear that (W_0, Y_{W_0}) is a V -module. The adjoint V -module (V, Y_V) containing the proper nontrivial V -submodule (W_0, Y_{W_0}) . But (V, Y_V) is not completely reducible since for any element $(u, v) \in V$ such that $u \neq 0$, $Y_V((0, \mathbf{1}), x)(u, v) = (0, u) \neq 0$ and is in W_0 .

The next result shows that the minimal Virasoro vertex operator algebras and their tensor product algebras satisfy, besides the rationality and the rationality of the gradings of irreducible modules, also the other conditions to use the tensor product theory:

Theorem 3.5 *For any positive integer m and any n pairs (p_i, q_i) of relatively prime positive integers larger than 1, $i = 1, \dots, m$, we have:*

1. Every finitely-generated lower-truncated generalized $L(c_{p_1, q_1}, 0) \otimes \cdots \otimes L(c_{p_m, q_m}, 0)$ -module is a module.
2. The products of the intertwining operators for

$$L(c_{p_1, q_1}, 0) \otimes \cdots \otimes L(c_{p_m, q_m}, 0)$$

have the convergence and extension property.

3. For any modules W_j , $j = 1, \dots, 2m + 1$, for

$$L(c_{p_1, q_1}, 0) \otimes \cdots \otimes L(c_{p_m, q_m}, 0),$$

any intertwining operators \mathcal{Y}_i , $i = 1, \dots, m$, of types $\left(\begin{smallmatrix} W_{2i-1} \\ W_{2i} W_{2i+1} \end{smallmatrix} \right)$, respectively, and any $w'_{(1)} \in W'_1$, $w_{(2i)} \in W_{2i}$, $i = 1, \dots, m$, and $w_{(2m+1)} \in W_{2m+1}$,

$$\left\langle w'_1, \mathcal{Y}_1(w_{(2)}, x_1) \cdots \mathcal{Y}_m(w_{(2m)}, x_m) w_{(2m+1)} \right\rangle \Big|_{x_i^n = e^{n \log z_i}, 1 \leq i \leq m, n \in \mathbb{C}}$$

is absolutely convergent for any $z_1, \dots, z_m \in \mathbb{C}$ satisfying $|z_1| > \cdots > |z_m| > 0$.

We now define precisely the class of vertex operator algebras that we study in this paper.

Definition 3.6 Let m be a nonnegative integer, (p_i, q_i) , $i = 1, \dots, m$, m pairs of relatively prime positive integers larger than 1. A vertex operator algebra V is said to be *in the class* $\mathcal{C}_{p_1, q_1; \dots; p_m, q_m}$ if V has a vertex operator subalgebra isomorphic to $L(c_{p_1, q_1}, 0) \otimes \cdots \otimes L(c_{p_m, q_m}, 0)$.

We have:

Proposition 3.7 *Let V be a vertex operator algebra in the class $\mathcal{C}_{p_1, q_1; \dots; p_m, q_m}$. Then every finitely-generated lower-truncated generalized V -module is a V -module.*

Combining Theorem 3.5 and Proposition 3.7 with Theorem 3.2, we obtain the main result of the present paper:

Theorem 3.8 *Let V be a vertex operator algebra in the class $\mathcal{C}_{p_1, q_1; \dots; p_m, q_m}$. Then for any V -modules W_1 and W_2 , $W_1 \boxtimes_{P(z)} W_2$ is a module and the conclusions of Proposition 1.4, Corollary 1.7, Theorem 1.9, Theorem 1.10, Theorem 3.1 and Theorem 1.11 are true.*

Combining this main result with Corollary 3.3, we obtain:

Corollary 3.9 *Let V be a vertex operator algebra in the class $\mathcal{C}_{p_1, q_1; \dots; p_m, q_m}$. Then the category of V -modules has a natural structure of a braided tensor category. In particular, the category of $L(c_{p_1, q_1}, 0) \otimes \dots \otimes L(c_{p_m, q_m}, 0)$ -modules has a natural structure of a braided tensor category.*

Let (p, q) be a pair of relatively prime positive integers larger than 1. We define the *category generated by minimal modules of central charge $c_{p, q}$ for the Virasoro algebra* to be the subcategory of the category of modules for the Virasoro algebra such that any object in this subcategory is isomorphic to a finite direct sum of $L(c_{p, q}, h_{p, q}^{m, n})$, $m, n \in \mathbb{Z}$, $0 < m < p$, $0 < n < q$. Then the special case $V = L(c_{p, q}, 0)$ in Corollary 3.9 can be reformulated as the following:

Theorem 3.10 *Let (p, q) be a pair of relatively prime positive integers larger than 1. Then the category generated by minimal modules of central charge $c_{p, q}$ for the Virasoro algebra has a natural structure of a braided tensor category.*

4 Proof of Theorem 3.1

By Theorem 1.10, there exist a V -module W_7 and intertwining operators \mathcal{Y}_5 and \mathcal{Y}_6 of type $\begin{pmatrix} W_7 \\ W_1 W_2 \end{pmatrix}$ and $\begin{pmatrix} W_4 \\ W_7 W_3 \end{pmatrix}$ such that for any $w'_{(4)} \in W'_4$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$ and for any $z_1, z_2 \in \mathbb{C}$ satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$,

$$\begin{aligned} & \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)} \rangle_{W_4} \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}} \\ &= \langle w'_{(4)}, \mathcal{Y}_6(\mathcal{Y}_5(w_{(1)}, x_0) w_{(2)}, x_2) w_{(3)} \rangle_{W_4} \Big|_{x_0^n = e^{n \log(z_1 - z_2)}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}}. \end{aligned} \tag{4.1}$$

Substituting

$$\begin{aligned}\mathcal{Y}_5(w_{(1)}, x_0)w_{(2)} &= \Omega_0(\Omega_1(\mathcal{Y}_5))(w_{(1)}, x_0)w_{(2)} \\ &= e^{x_0 L(-1)}\Omega_1(\mathcal{Y}_5)(w_{(2)}, e^{\pi i}x_0)w_{(1)}.\end{aligned}$$

into (4.1), we obtain

$$\begin{aligned}&\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)} \rangle_{W_4} \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}} \\ &= \langle w'_{(4)}, \mathcal{Y}_6(e^{x_0 L(-1)}\Omega_1(\mathcal{Y}_5)(w_{(2)}, e^{\pi i}x_0)w_{(1)}, x_2) \cdot \\ &\quad \cdot w_{(3)} \rangle_{W_4} \Big|_{x_0^n = e^{n \log(z_1 - z_2)}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}} \\ &= \langle w'_{(4)}, \mathcal{Y}_6(\Omega_1(\mathcal{Y}_5)(w_{(2)}, e^{\pi i}x_0)w_{(1)}, x_2 + x_0) \cdot \\ &\quad \cdot w_{(3)} \rangle_{W_4} \Big|_{x_0^n = e^{n \log(z_1 - z_2)}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}}\end{aligned}\tag{4.2}$$

when $|z_1| > |z_2| > |z_1 - z_2| > 0$. The right-hand side of (4.2) is the analytic extension of

$$\langle w'_{(4)}, \mathcal{Y}_6(\Omega_1(\mathcal{Y}_5)(w_{(2)}, x_0)w_{(1)}, x_1)w_{(3)} \rangle_{W_4} \Big|_{x_0^n = e^{n \log(z_2 - z_1)}, x_1^n = e^{n \log z_1}, n \in \mathbb{C}}$$

defined in the region $|z_1| > |z_1 - z_1| > 0$. By Theorem 1.10, there exist a V -module W_6 and intertwining operators \mathcal{Y}_3 and \mathcal{Y}_4 of type $\begin{pmatrix} W_4 \\ W_2 W_6 \end{pmatrix}$ and $\begin{pmatrix} W_6 \\ W_1 W_3 \end{pmatrix}$, respectively, such that for any $z_1, z_2 \in \mathbb{C}$ satisfying $|z_2| > |z_1| > |z_1 - z_2| > 0$,

$$\begin{aligned}&\langle w'_{(4)}, \mathcal{Y}_6(\Omega_1(\mathcal{Y}_5)(w_{(2)}, x_0)w_{(1)}, x_1)w_{(3)} \rangle_{W_4} \Big|_{x_0^n = e^{n \log(z_2 - z_1)}, x_1^n = e^{n \log z_1}, n \in \mathbb{C}} \\ &= \langle w'_{(4)}, \mathcal{Y}_3(w_{(2)}, x_2)\mathcal{Y}_4(w_{(1)}, x_1)w_{(3)} \rangle_{W_4} \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}}.\end{aligned}\tag{4.3}$$

Thus the left-hand side of (4.1) and the right-hand side of (4.3) are analytic extensions of each other, proving the theorem.

5 Proof of Theorem 3.2

Let V be a vertex operator algebra containing a rational vertex operator subalgebra V_0 . We shall denote the $P(z)$ -tensor product operations for V

and for V_0 by $\boxtimes_{P(z)}$ and $\boxtimes_{P(z)}^{V_0}$, respectively. Similarly, we have the notations $\boxtimes_{P(z)}$ and $\boxtimes_{P(z)}^{V_0}$. Let W_1 and W_2 be two V -module. Then W_1 and W_2 are also V_0 -modules and by definition we have $W_1 \boxtimes_{P(z)} W_2 \subset W_1 \boxtimes_{P(z)}^{V_0} W_2$. Since V_0 is rational, $W_1 \boxtimes_{P(z)}^{V_0} W_2$ is a V_0 -module. In particular, its homogeneous subspaces are finite-dimensional and when the real part of the weight of a homogeneous subspace is sufficiently small, the homogeneous subspace is 0. Thus $W_1 \boxtimes_{P(z)} W_2$ also satisfies these two conditions. This shows that $W_1 \boxtimes_{P(z)} W_2$ is a V -module. By Proposition 1.4 and Corollary 1.7, the conclusions of Proposition 1.4 and Corollary 1.7 are true.

Next we assume in addition that all irreducible V_0 -modules are \mathbb{R} -graded and V and V_0 satisfy Condition 1 and Condition 2 in Theorem 1.9. Since any irreducible V -module is a V_0 -module and by assumption any irreducible V_0 -module, thus also any V_0 -module, is \mathbb{R} -graded, we see that any irreducible V -module is \mathbb{R} -graded. In the proofs of Theorem 1.9 and Theorem 1.10 in [H1] and thus also in the proof of Theorem 3.1 in the preceding section, the rationality is used only in the proof of Lemma 14.4 and in Remark 16.1 in [H1]. So to prove the conclusions of Theorem 1.9, Theorem 1.10 and Theorem 3.1 in this case, we need first to show that the conclusions of Lemma 14.4 and Remark 16.1 in [H1] are still true in this case. It is clear that the conclusions of Lemma 14.4 in [H1] are still true since \mathcal{Y} is also an intertwining operators for V_0 of type $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$ if W_1 , W_2 and W_3 are viewed as V_0 -modules and thus Lemma 14.4 in [H1] can be used. Conclusions of Remark 16.1 in [H1] are also true in this case since they are obtained by comparing weights and coefficients of series and thus Remark 16.1 for intertwining operators for V_0 can be used to obtain the conclusions in this case. To show that the conclusions of Theorem 1.9, Theorem 1.10 and Theorem 3.1 are true in this case, we still need to show that V satisfies Condition 1 and Condition 2 in Theorem 1.9. Condition 1 is satisfied by assumption. Since any intertwining operator for V are also intertwining operators for V_0 when V -modules are viewed as V_0 -modules, Condition 2 is also satisfied.

Finally we assume in addition that V_0 satisfies Condition 3 in Theorem 3.2. Since any intertwining operator for V are also intertwining operators for V_0 when V -modules are viewed as V_0 -modules, Condition 2 in Theorem 1.9 and Condition 3 in Theorem 1.11 are satisfied. Condition 1 in Theorem 1.9 is satisfied by assumption. We already know that any irreducible V -module is \mathbb{R} -graded. As in the proofs of Theorem 1.9, Theorem 1.10 and Theorem

3.1, in the proof of Theorem 1.11, the rationality of V is also used only in the proof of Lemma 14.4 and in Remark 16.1 in [H1]. So the conclusion of Theorem 1.11 is true in this case.

6 Proof of Theorem 3.5

We first prove the case with $m = 1$, that is, the case for minimal Virasoro vertex operator algebras.

Let W be a lower-truncated generalized $L(c_{p,q}, 0)$ -module generated by a homogeneous vector $w \in W$. Then any element of W is a linear combination of the elements

$$L(n_1) \cdots L(n_k)w, \quad k \in \mathbb{N}, \quad n_1, \dots, n_k \in \mathbb{Z}. \quad (6.1)$$

Using the Virasoro commutator relations for the operators $L(n)$, $n \in \mathbb{Z}$, any element of the form (6.1), and thus any element of W , can be expressed as linear combinations of the elements

$$L(-m_1) \cdots L(-m_k)L(n_1) \cdots L(n_l)w, \quad k, l \in \mathbb{N}, \quad m_1, \dots, m_k, n_1, \dots, n_l \in \mathbb{Z}_+. \quad (6.2)$$

Since W is lower-truncated, we see that for any fixed complex number, there are only finitely many elements of the form (6.2) with weight equal to this complex number. Thus the homogeneous subspaces of W are all finite-dimensional, proving that W is a module. The first conclusion is now an immediate consequence.

We prove the second conclusion using the differential equations of Belavin, Polyakov and Zamolodchikov (BPZ equations) for correlation functions in the minimal models [BPZ] and the theory of differential equations of regular singular points. It is clear that we need only to prove the case that the intertwining operators are among irreducible $L(c_{p,q}, 0)$ -modules. Given any irreducible $L(c_{p,q}, 0)$ -modules W_1, W_2, W_3, W_4 and W_5 , let $\mathcal{Y}_1, \mathcal{Y}_2$, be intertwining operators of type $\begin{pmatrix} W_4 \\ W_1 W_5 \end{pmatrix}, \begin{pmatrix} W_5 \\ W_2 W_3 \end{pmatrix}$, respectively and $w_{(1)}, w_{(2)}, w_{(3)}$ and $w'_{(4)}$ the lowest weight vectors of W_1, W_2, W_3 and W'_4 , respectively. We first would like to show that when $|z_1| > |z_2| > 0$,

$$\langle w'_{(4)} \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)} \rangle \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}} \quad (6.3)$$

satisfies a BPZ equation. From the representation theory of the Virasoro algebra, we know that there exists

$$P = \sum_{i=1}^k a_i L(-m_1^{(i)}) \cdots L(-m_{l_i}^{(i)}) \in U(\mathfrak{L}_-) \quad (6.4)$$

where $l_i, m_j^{(i)} \in \mathbb{Z}_+$, $1 \leq j \leq l_i$, $1 \leq i \leq k$ such that

$$\sum_{j=1}^{l_1} m_j^{(1)} = \cdots = \sum_{j=1}^{l_k} m_j^{(k)} > 0. \quad (6.5)$$

and

$$Pw_{(3)} = 0.$$

Thus we have

$$\langle w'_{(4)} \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) Pw_{(3)} \rangle \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}} = 0. \quad (6.6)$$

Since $w_{(1)}$, $w_{(2)}$ and $w'_{(4)}$ are lowest weight vectors,

$$\begin{aligned} & \langle w'_{(4)} \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) L(-m) w \rangle \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}} = \\ & = -\langle w'_{(4)} (x_1^{-m+1} \frac{\partial}{\partial x_1} + (\text{wt } w_{(1)}) (-m+1) x_1^{-m}) \cdot \\ & \quad \cdot \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) w \rangle \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}} \\ & \quad - \langle w'_{(4)} \mathcal{Y}_1(w_{(1)}, x_1) (x_2^{-m+1} \frac{\partial}{\partial x_2} + (\text{wt } w_{(2)}) (-m+1) x_2^{-m}) \cdot \\ & \quad \cdot \mathcal{Y}_2(w_{(2)}, x_2) w \rangle \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}} \\ & = -(z_1^{-m+1} \frac{\partial}{\partial z_1} + (\text{wt } w_{(1)}) (-m+1) z_1^{-m} \\ & \quad + (z_2^{-m+1} \frac{\partial}{\partial z_2} + (\text{wt } w_{(2)}) (-m+1) z_2^{-m})) \cdot \\ & \quad \cdot \langle w'_{(4)} \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) w \rangle \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}} \quad (6.7) \end{aligned}$$

for any $w \in W_{(3)}$ and $m \in \mathbb{Z}_+$. From (6.4), (6.6) and (6.7), we see that (6.3) satisfies a differential equation. Similarly there exists

$$Q = \sum_{i=1}^r b_i L(-n_1^{(i)}) \cdots L(-n_{s_i}^{(i)}) \in U(\mathfrak{L}_-) \quad (6.8)$$

where $s_i, n_j^{(i)} \in \mathbb{Z}_+$, $1 \leq j \leq s_i$, $1 \leq i \leq r$ such that

$$\sum_{j=1}^{s_1} n_j^{(1)} = \cdots = \sum_{j=1}^{s_r} n_j^{(r)} > 0. \quad (6.9)$$

and

$$Qw_{(2)} = 0.$$

Thus we have

$$\langle w'_{(4)} \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(Qw_{(2)}, x_2) w_{(3)} \rangle \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}} = 0. \quad (6.10)$$

Recalling that we have a linear isomorphism

$$\Omega_{-1} : \mathcal{V}_{W_2 W_3}^{W_5} \rightarrow \mathcal{V}_{W_3 W_2}^{W_5}$$

and its inverse

$$\Omega_0 : \mathcal{V}_{W_3 W_2}^{W_5} \rightarrow \mathcal{V}_{W_2 W_3}^{W_5}$$

defined by

$$\Omega_0(\mathcal{Y}(w_{(3)}, x)w_{(2)}) = e^{xL(-1)} \mathcal{Y}(w_{(2)}, e^{\pi i} x)w_{(3)}$$

(see [FHL] and [HL3]), we obtain

$$\begin{aligned} & \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(Qw_{(2)}, x_2) w_{(3)} \rangle \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}} = \\ &= \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \Omega_0(\Omega_{-1}(\mathcal{Y}_2))(Qw_{(2)}, x_2) w_{(3)} \rangle \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}} \\ &= \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) e^{x_2 L(-1)} \cdot \\ & \quad \cdot \Omega_{-1}(\mathcal{Y}_2)(w_{(3)}, e^{\pi i} x_2) Qw_{(2)} \rangle \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}} \\ &= \langle e^{x_2 L(1)} w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1 - x_2) \cdot \end{aligned}$$

$$\begin{aligned}
& \cdot \Omega_{-1}(\mathcal{Y}_2)(w_{(3)}, e^{\pi i} x_2) Q w_{(2)} \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}} \\
&= \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1 - x_2) \\
& \quad \cdot \Omega_{-1}(\mathcal{Y}_2)(w_{(3)}, e^{\pi i} x_2) Q w_{(2)} \rangle \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}}. \tag{6.11}
\end{aligned}$$

Using (6.8), (6.10) and (6.11) and the same method as above, we obtain another differential equation satisfied by (6.3). From (6.7) and (6.11), we see that $z_1 = z_2$ are not singularities of the differential equation obtained from (6.6) but are singularities of the differential equation obtained from (6.10). So the two differential equations are independent. We obtain a system of two independent differential equations for functions with two variables. Formally, (6.3) satisfies this system of equations. From (6.5), (6.7), (6.9) and (6.11), we see that this system has only regular singularities $z_1, z_2 = 0, \infty$ and $z_1 = z_2$. Since \mathcal{Y}_1 and \mathcal{Y}_2 are intertwining operators among irreducible modules, there exist rational numbers h_1 and h_2 such that

$$\mathcal{Y}_1(w_{(1)}, x_1) \in x^{h_1} \text{Hom}(W_5, W_4)[[x_1, x_1^{-1}]]$$

and

$$\mathcal{Y}_2(w_{(2)}, x_2) \in x^{h_2} \text{Hom}(W_3, W_5)[[x_2, x_2^{-1}]].$$

Thus by the definition of intertwining operator, there are two rational numbers t_1, t_2 and a power series $g(z)$ such that (6.3) is equal to

$$z_1^{t_1} z_2^{t_2} g(z_2/z_1). \tag{6.12}$$

The system of differential equations for (6.3) give a differential equation for $g(z)$. Since the system of differential equations for (6.3) only have singularity when $z_1, z_2 = 0, \infty$ and $z_1 = z_2$ and $g(z)$ is a power series, this equation for $g(z)$ has coefficients analytic in $|z| < 1$. The coefficients of $g(z)$ are can be calculated using $\mathcal{Y}_1, \mathcal{Y}_2, w_{(1)}, w_{(2)}, w_{(3)}$ and $w'_{(4)}$ and thus the initial conditions at $z = 0$ for the initial-value problem for the differential equation for $g(z)$ are known. By the existence and uniqueness theorem of differential equations with analytic coefficients, $g(z)$ must be convergent when $|z| < 1$. Thus (6.12) and (6.3) are convergent when $|z_2| > |z_1| > 0$.

Now we can use the values of (6.3) and its derivatives at (z_1, z_2) satisfying $|z_2| > |z_1| > |z_1 - z_2| > 0$ as the initial condition to solve the system of

equations of regular singular points for (6.3) in the region $|z_2| > |z_1 - z_2| > 0$. By the theory of equations of regular singular points (see, for example, Appendix B of [K]), there exist $j \in \mathbb{N}$, $r_i, s_i \in \mathbb{C}$, $i = 1, \dots, j$, $M \in \mathbb{N}$ and analytic functions $f_{i,m_1,m_2}(z)$ on $|z| < 1$, $i = 1, \dots, j$, $m_1, m_2 \in \mathbb{N}$, $m_1 + m_2 \leq M$, such that the solution in this region is of the form

$$\sum_{i=1}^j \sum_{m_1, m_2 \in \mathbb{N}, m_1 + m_2 \leq M} z_2^{r_i} (z_1 - z_2)^{s_i} (\log z_2)^{m_1} \cdot \left(\log \left(\frac{z_1 - z_2}{z_2} \right) \right)^{m_2} f_{i,m_1,m_2} \left(\frac{z_1 - z_2}{z_2} \right). \quad (6.13)$$

Since (6.13) is the analytic extension of (6.3) and thus also of (6.12) and since (6.12) at the singularity $z_2 = 0$ does not contain any term involving $\log z_2$, we have

$$f_{i,m_1,m_2}(z) = 0$$

for any $i = 1, \dots, j$ and any z satisfying $|z| < 1$ if m_1 or m_2 is not 0. Comparing (6.13) with (6.12), we also see that we can take r_i and s_i , $i = 1, \dots, j$, to be rational numbers. Thus (6.13) or the analytic extension of (6.3) in the region $|z_2| > |z_1 - z_2| > 0$ must be of the form (1.4).

Let N be an integer such that for this extension and $w_{(1)}$ and $w_{(2)}$ above, (1.3) holds. We now use induction on the weights of $w_{(1)}$, $w_{(2)}$, $w_{(3)}$ and $w'_{(4)}$ to show that for any $w_{(1)}$, $w_{(2)}$, $w_{(3)}$ and $w'_{(4)}$, not necessarily lowest weight vectors, (6.3) is convergent when $|z_1| > |z_2| > 0$ and can be analytically extended to a function of the form (1.4) in the region $|z_2| > |z_1 - z_2| > 0$ satisfying (1.3) for the N above. First we assume that $w_{(1)}$ and $w_{(2)}$ are still lowest weight vectors. In this case, instead of (6.7), we have

$$\begin{aligned} & \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) L(-m)w \rangle \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}} \\ &= \langle L(m)w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2)w \rangle \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}} \\ & \quad - (z_1^{-m+1} \frac{\partial}{\partial z_1} + (\text{wt } w_{(1)})(-m+1)z_1^{-m} \\ & \quad + z_2^{-m+1} \frac{\partial}{\partial z_2} + (\text{wt } w_{(2)})(-m+1)z_2^{-m}) \cdot \\ & \quad \cdot \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2)w \rangle \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}}. \quad (6.14) \end{aligned}$$

Since the weight of $L(m)w'_{(4)}$ is less than the weight of $w'_{(4)}$ and the weight of w is less than the weight of $L(-m)w$, using induction we see that the left-hand side of (6.14) is convergent when $|z_1| > |z_2| > 0$ and can be analytically extended to a function of the form (1.4) in the region $|z_2| > |z_1 - z_2| > 0$. Also from (6.14), we see that if the analytic extensions of

$$\langle L(m)w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) w \rangle \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}}$$

and

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) w \rangle \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}}$$

in the region $|z_2| > |z_1 - z_2| > 0$ are of the form (1.4) satisfying (1.3), the extension of the left-hand side of (6.14) can also be written in the form (1.4) satisfying (1.3). Similarly we can prove the case for $w'_{(4)} = L'(-m)w'$ using the case for w' .

Next we consider the case that only $w_{(1)}$ is still a lowest weight vector. By the Jacobi identity and the definition of contragredient vertex operator, we have

$$\begin{aligned} & \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(L(-m)w, x_2) w_{(3)} \rangle \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}} \\ &= -\text{Res}_x (-x + x_2)^{-m+1} \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \cdot \\ & \quad \cdot \mathcal{Y}_2(w, x) Y_3(\omega, x_2) w_{(3)} \rangle \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}} \\ &+ \text{Res}_x (x_2 - x)^{-m+1} \langle w'_{(4)}, Y_4(\omega, x) \cdot \\ & \quad \cdot \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w, x) w_{(3)} \rangle \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}} \\ &- \text{Res}_x (x - x_2)^{-m+1} \text{Res}_{x_0} x_0^{-1} \delta \left(\frac{x - x_0}{x_1} \right) \langle w'_{(4)}, \mathcal{Y}_1(Y_1(\omega, x_0) w_{(1)}, x_1) \cdot \\ & \quad \cdot \mathcal{Y}_2(w, x_2) w_{(3)} \rangle \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}} \\ &= -\text{Res}_x (-x_2 + x)^{-m+1} \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \cdot \\ & \quad \cdot \mathcal{Y}_2(w, x_2) Y_3(\omega, x) w_{(3)} \rangle \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}} \\ &+ \text{Res}_x (x - x_2)^{-m+1} x^{-2} \langle Y'_4(\omega, x^{-1}) w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \cdot \end{aligned}$$

$$\begin{aligned}
& \cdot \mathcal{Y}_2(w, x)w_{(3)} \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}} \\
& - \text{Res}_x (x - x_2)^{-m+1} \text{Res}_{x_0} x_0^{-1} \delta \left(\frac{x - x_0}{x_1} \right) \langle w'_{(4)}, \mathcal{Y}_1((x_0^{-2} L(0) \\
& + x_0^{-1} L(-1))w_{(1)}, x_1) \mathcal{Y}_2(w, x_2)w_{(3)} \rangle \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}} \\
& = - \text{Res}_x (-x_2 + x)^{-m+1} \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \cdot \\
& \cdot \mathcal{Y}_2(w, x_2)Y_3(\omega, x)w_{(3)} \rangle \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}} \\
& + \text{Res}_x (x - x_2)^{-m+1} x^{-2} \langle Y'_4(\omega, x^{-1})w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \cdot \\
& \cdot \mathcal{Y}_2(w, x)w_{(3)} \rangle \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}} \\
& - (\text{wt } w_{(1)}) \text{Res}_x (x - x_2)^{-m+1} \text{Res}_{x_0} x_0^{-3} \delta \left(\frac{x - x_0}{x_1} \right) \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \cdot \\
& \cdot \mathcal{Y}_2(w, x_2)w_{(3)} \rangle \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}} \\
& - \text{Res}_x (x - x_2)^{-m+1} \text{Res}_{x_0} x_0^{-2} \delta \left(\frac{x - x_0}{x_1} \right) \frac{\partial}{\partial x_1} \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \cdot \\
& \cdot \mathcal{Y}_2(w, x_2)w_{(3)} \rangle \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}} \tag{6.15}
\end{aligned}$$

for any $w \in W_2$ and $m \in \mathbb{Z}_+$. Using induction on the weight of $w_{(2)}$, we see from (6.15) that the left-hand side of (6.15) is convergent when $|z_1| > |z_2| > 0$ and can be analytically extended to a function of the form (1.4) in the region $|z_2| > |z_1 - z_2| > 0$ satisfying (1.3). The most general case that $w_{(1)}, w_{(2)}, w_{(3)}, w'_{(4)}$ are arbitrary can be proved similarly using a formula a little more complicated than (6.1) and induction on the weight of $w_{(1)}$.

The proof of the third conclusion in this special case is similar to the proof of the convergence in the proof of the second conclusion above by deriving certain BPZ equations.

To prove the general case, we need:

Lemma 6.1 *Let m be a positive integer, (p_i, q_i) , $i = 1, \dots, m$, m pairs of relatively prime positive integers larger than 1, $V = L(c_{p_1, q_1}, 0) \otimes \dots \otimes L(c_{p_m, q_m}, 0)$, $W_i = L(c_{p_1, q_1}, h_1^{(t)}) \otimes \dots \otimes L(c_{p_m, q_m}, h_m^{(t)})$, $t = 1, 2, 3$, irreducible V -modules and \mathcal{Y} an intertwining operator of type $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$. Then there exist*

intertwining operators \mathcal{Y}_i of type $\left(\begin{smallmatrix} L(c_{p_i, q_i}, h_i^{(3)}) \\ L(c_{p_i, q_i}, h_i^{(1)}) L(c_{p_i, q_i}, h_i^{(2)}) \end{smallmatrix} \right)$, $i = 1, \dots, m$, such that

$$\mathcal{Y} = \mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_m. \quad (6.16)$$

Proof Since $L(c_{p_i, q_i}, 0)$, $i = 1, \dots, m$, are rational, by Proposition 2.10 in [DMZ], we know that the vector space $\mathcal{V}_{W_1 W_2}^{W_3}$ is isomorphic the vector space

$$\bigotimes_{i=1}^m \mathcal{V}_{L(c_{p_i, q_i}, h_i^{(1)}) L(c_{p_i, q_i}, h_i^{(2)})}^{L(c_{p_i, q_i}, h_i^{(3)})}. \quad (6.17)$$

On the other hand, intertwining operators of the form (6.16) form a subspace of $\mathcal{V}_{W_1 W_2}^{W_3}$ with dimension equal to the dimension of (6.17). So this subspace must equal to $\mathcal{V}_{W_1 W_2}^{W_3}$ itself, proving the lemma. This lemma can also be proved directly using the special properties of the Virasoro vertex operator algebras. \square

Now we prove the theorem in the general case.

Since $L(c_{p_i, q_i}, 0)$, $i = 1, \dots, m$, are rational, $V = L(c_{p_1, q_1}, 0) \otimes \cdots \otimes L(c_{p_m, q_m}, 0)$ is also rational [DMZ]. Since for any $i \in \mathbb{Z}$, $1 \leq i \leq m$, every finitely-generated lower-truncated generalized $L(c_{p_i, q_i}, 0)$ -module is a module, the same method proving the rationality of $V = L(c_{p_1, q_1}, 0) \otimes \cdots \otimes L(c_{p_m, q_m}, 0)$ shows that any finitely-generated lower-truncated generalized V -module W is a sum of V -modules. Since there are only finitely many irreducible V -modules and W is finitely-generated, W must be an V -module.

For simplicity we only prove the second conclusion for $m = 2$. Consider intertwining operators \mathcal{Y}_1 and \mathcal{Y}_2 of type $\left(\begin{smallmatrix} W_4 \\ W_1 W_5 \end{smallmatrix} \right)$ and $\left(\begin{smallmatrix} W_5 \\ W_2 W_3 \end{smallmatrix} \right)$, respectively. For simplicity, we assume that W_1, \dots, W_5 are irreducible modules. Thus we have $W_i = L(c_{p_1, q_1}, h_1^{(t)}) \otimes L(c_{p_2, q_2}, h_2^{(t)})$, $t = 1, \dots, 5$. By Lemma 6.1, there are intertwining operators $\mathcal{Y}_1^{(i)}$ and $\mathcal{Y}_1^{(i)}$, $i = 1, 2$, of type $\left(\begin{smallmatrix} L(c_{p_i, q_i}, h_i^{(5)}) \\ L(c_{p_i, q_i}, h_i^{(1)}) L(c_{p_i, q_i}, h_i^{(4)}) \end{smallmatrix} \right)$ and $\left(\begin{smallmatrix} L(c_{p_i, q_i}, h_i^{(4)}) \\ L(c_{p_i, q_i}, h_i^{(2)}) L(c_{p_i, q_i}, h_i^{(3)}) \end{smallmatrix} \right)$, respectively, such that $\mathcal{Y}_j = \mathcal{Y}_j^{(1)} \otimes \mathcal{Y}_j^{(2)}$, $j = 1, 2$. By the case with $m = 1$, the products of the intertwining operators for $L(c_{p_i, q_i}, 0)$, $i = 1, 2$, have the convergence and extension property. Let N_1 and N_2 be the integers for $\mathcal{Y}_1^{(1)}, \mathcal{Y}_2^{(1)}$ and $\mathcal{Y}_1^{(2)}, \mathcal{Y}_2^{(2)}$, respectively, in the convergence and extension property for products. Then for any $w_{(l)}^{(i)} \in L(c_{p_i, q_i}, h_i^{(t)})$, $i = 1, 2$, $t = 1, 2, 3$ and $(w_{(4)}^{(i)})' \in L(c_{p_i, q_i}, h_i^{(4)})'$, $i = 1, 2$, there exist rational

numbers $r_j^{(i)}, s_j^{(i)}$ and analytic functions $f_j^{(i)}(z)$ on $|z| < 1$, $j = 1, \dots, k_i$, $i = 1, 2$, satisfying $\text{wt } w_{(1)}^{(i)} + \text{wt } w_{(2)}^{(i)} + s_j^{(i)} > N_i$ such that

$$\langle (w_{(4)}^{(i)})', \mathcal{Y}_1^{(i)}(w_{(1)}^{(i)}, x_1)(\mathcal{Y}_2^{(i)}(w_{(2)}^{(i)}, x_2)w_{(3)}^{(i)}) \rangle_{L(c_{p(i)}, q_{(i)}, h_{(i)}^{(4)})} \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{Q}}$$

is absolutely convergent when $|z_1| > |z_2| > 0$ and can be analytically extended to

$$\sum_{j=1}^k z_2^{r_j^{(i)}} (z_1 - z_2)^{s_j^{(i)}} f_j^{(i)} \left(\frac{z_1 - z_2}{z_2} \right)$$

when $|z_2| > |z_1 - z_2| > 0$. Thus

$$\begin{aligned} & \langle (w_{(4)}^{(1)})' \otimes (w_{(4)}^{(2)})', \mathcal{Y}_{(1)}(w_{(1)}^{(1)} \otimes w_{(1)}^{(2)}, x_1) \cdot \\ & \quad \cdot \mathcal{Y}_{(2)}(w_{(2)}^{(1)} \otimes w_{(2)}^{(2)}, x_2)(w_{(3)}^{(1)} w_{(3)}^{(2)}) \rangle_{W_4} \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{Q}} \\ &= \langle (w_{(4)}^{(1)})', \mathcal{Y}_1^{(1)}(w_{(1)}^{(1)}, x_1) \cdot \\ & \quad \cdot \mathcal{Y}_2^{(1)}(w_{(2)}^{(1)}, x_2)w_{(3)}^{(1)} \rangle_{L(c_{p1}, q_1, h_1^{(4)})} \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{Q}} \\ & \quad \cdot \langle (w_{(4)}^{(2)})', \mathcal{Y}_1^{(2)}(w_{(1)}^{(2)}, x_1) \cdot \\ & \quad \cdot \mathcal{Y}_2^{(2)}(w_{(2)}^{(2)}, x_2)w_{(3)}^{(2)} \rangle_{L(c_{p2}, q_2, h_2^{(4)})} \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{Q}} \end{aligned}$$

is absolutely convergent and can be analytic extended to

$$\begin{aligned} & \sum_{j_1=1}^{k_1} z_2^{r_{j_1}^{(1)}} (z_1 - z_2)^{s_{j_1}^{(1)}} f_{j_1}^{(1)} \left(\frac{z_1 - z_2}{z_2} \right) \cdot \\ & \quad \cdot \sum_{j_2=1}^{k_2} z_2^{r_{j_2}^{(2)}} (z_1 - z_2)^{s_{j_2}^{(2)}} f_{j_2}^{(2)} \left(\frac{z_1 - z_2}{z_2} \right) \\ &= \sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} z_2^{(r_{j_1}^{(1)} + r_{j_2}^{(2)})} (z_1 - z_2)^{(s_{j_1}^{(1)} + s_{j_2}^{(2)})} f_{j_1}^{(1)} \left(\frac{z_1 - z_2}{z_2} \right) f_{j_2}^{(2)} \left(\frac{z_1 - z_2}{z_2} \right). \end{aligned}$$

And we have

$$\begin{aligned} & \text{wt } (w_{(1)}^{(1)} \otimes w_{(1)}^{(2)}) + \text{wt } (w_{(2)}^{(1)} \otimes w_{(2)}^{(2)}) + (s_{j_1}^{(1)} + s_{j_2}^{(2)}) \\ &= (\text{wt } w_{(1)}^{(1)} + \text{wt } w_{(2)}^{(1)} + s_{j_1}^{(1)}) + (\text{wt } w_{(1)}^{(2)} + \text{wt } w_{(2)}^{(2)} + s_{j_2}^{(2)}) \\ &> N_1 + N_2, \end{aligned}$$

$j_1 = 1, \dots, k_1, j_2 = 1, \dots, k_2$.

The proof of the third conclusion is similar to the proof of the convergence in the proof of the second conclusion above.

7 Proof of Proposition 3.7

We still only prove the result for $m = 2$. Since V has a subalgebra isomorphic to $L(c_{p_1, q_1}, 0) \otimes L(c_{p_2, q_2}, 0)$, any V -module is a $L(c_{p_1, q_1}, 0) \otimes L(c_{p_2, q_2}, 0)$ -module. In particular, V is an $L(c_{p_1, q_1}, 0) \otimes L(c_{p_2, q_2}, 0)$ -module. Since $L(c_{p_1, q_1}, 0) \otimes L(c_{p_2, q_2}, 0)$ is rational, V can be decomposed as a direct sum of irreducible $L(c_{p_1, q_1}, 0) \otimes L(c_{p_2, q_2}, 0)$ -modules. Thus by Theorem 4.7.4 of [FHL], as a $L(c_{p_1, q_1}, 0) \otimes L(c_{p_2, q_2}, 0)$ -module

$$V = \sum_{j=1}^k L(c_{p_1, q_1}, h_1^{(j)}) \otimes L(c_{p_2, q_2}, h_2^{(j)}).$$

Let W be a lower-truncated generalized V -module generated by one homogeneous element $w \in W$. Then it is a generalized $L(c_{p_1, q_1}, 0) \otimes L(c_{p_2, q_2}, 0)$ -module. By a lemma of Dong-Mason [DM2] and Li [L], W is spanned by elements of the form $v_n w$ where $v \in V, n \in \mathbb{C}$. Let $u_{(j)}^{(i)}, i = 1, 2, j = 1, \dots, k$, be the lowest weight vectors of $L(c_{p, q}, h_i^{(j)})$, respectively. Using the Jacobi identity for generalized modules, we see that elements of the form $v_n w$ are spanned by elements of the form

$$\begin{aligned} & (L(-m_1^{(1)}) \cdots L(-m_{p_1}^{(1)}) \otimes L(-m_1^{(2)}) \cdots L(-m_{p_2}^{(2)})) \cdot \\ & \cdot (L(-1)^{l_1} u_{(j_1)}^{(1)})_{j_1} \otimes (L(-1)^{l_2} u_{(j_2)}^{(2)})_{j_2} \cdot \\ & \cdot (L(n_1^{(1)}) \cdots L(n_{q_1}^{(1)}) \otimes L(n_1^{(2)}) \cdots L(n_{q_2}^{(2)})) w, \end{aligned} \quad (7.1)$$

$m_1^{(1)}, \dots, m_{p_1}^{(1)}, m_1^{(2)}, \dots, m_{p_2}^{(2)}, n_1^{(1)}, \dots, n_{p_1}^{(1)}, n_1^{(2)}, \dots, n_{p_2}^{(2)} \in \mathbb{Z}_+, l_1, l_2 \in \mathbb{N}, j_1, j_2 \in \mathbb{Q}, j = 1, \dots, k$. By Theorem 3.5, the generalized $L(c_{p_1, q_1}, 0) \otimes L(c_{p_2, q_2}, 0)$ -module generated by w is a module and thus a finite direct sum of irreducible $L(c_{p_1, q_1}, 0) \otimes L(c_{p_2, q_2}, 0)$ -modules. Thus by Theorem 4.7.4 of [FHL], $w = \sum_{t=1}^r w_{(t)}^{(1)} \otimes w_{(t)}^{(2)}$ where $w_{(t)}^{(i)}, t = 1, \dots, r, i = 1, 2$, are homogeneous elements of irreducible $L(c_{p_i, q_i}, 0)$ -modules. Thus elements of the form (7.1)

are spanned by elements of the form

$$L(-m_1^{(1)}) \cdots L(-m_{p_1}^{(1)})(L(-1)^{l_1} u_{(j)}^{(1)})_{j_1} L(n_1^{(1)}) \cdots L(n_{q_1}^{(1)}) w_{(t)}^1 \otimes \\ \otimes L(-m_1^{(2)}) \cdots L(-m_{p_2}^{(2)})(L(-1)^{l_2} u_{(j)}^{(2)})_{j_2} L(n_1^{(2)}) \cdots L(n_{q_2}^{(2)}) w_{(t)}^2, ,$$

$m_1^{(1)}, \dots, m_{p_1}^{(1)}, m_1^{(2)}, \dots, m_{p_2}^{(2)}, n_1^{(1)}, \dots, n_{p_1}^{(1)}, n_1^{(2)}, \dots, n_{p_2}^{(2)} \in \mathbb{Z}_+, l_1, l_2 \in \mathbb{N}, j_1, j_2 \in \mathbb{Q}, t = 1, \dots, r, j = 1, \dots, k$. Using the $L(-1)$ -derivative property for generalized modules, we see that elements of the form (7.2) are spanned by elements of the form (7.2) with $l_1 = l_2 = 0$. Now consider the elements of the form (7.2) with $l_1 = l_2 = 0$ and of a fixed weight s . Since W is lower-truncated, there are only finitely many of them. This proves that the homogeneous subspaces of W are finite-dimensional. So W is a V -module.

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